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On waves generated in dispersive systems by travelling forcing effects, with applications to the dynamics of rotating fluids

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A theory of the generation of dispersive waves by travelling forcing effects, that may be steady, oscillatory or transient in character, is given for a general homogeneous system. Small disturbances to the system are supposed stable, and governed by a linear partial differential equation with constant coefficients which admits solutions in the form of plane waves satisfying an, in general, anisotropic dispersion relation $P(\sigma, \mathbf{k}) = 0$ between frequency σ and wave-number vector $\mathbf{k}$.

If the forcing region, supposed of limited extent, travels with constant velocity $\mathbf{U}$, then oscillatory forcing terms of frequency σ_0 (which would be replaced by 0 in the limiting case of a steady forcing effect, while taking, for a typical transient one of duration T, values from 0 to about $10/T$) produce waves of frequency $\sigma_0 + \mathbf{U}.\mathbf{k}$ (the Doppler effect). For any such waves, the wave-number $\mathbf{k}$ satisfies the equation $P(\sigma_0 + \mathbf{U}.\mathbf{k}, \mathbf{k}) = 0$, representing a surface in wave-number space here called $S(\sigma_0)$, and their position relative to that of the forcing region is determined by having been generated when that region was in an earlier position, and having subsequently progressed with the group velocity. This implies the rule, also derived analytically in §§2 and 3, that waves with a particular value of $\mathbf{k}$ on $S(\sigma_0)$ are found in a direction, stretching out from the forcing region, which is one of the directions normal to $S(\sigma_0)$ at $\mathbf{k}$, namely the one pointing towards $S(\sigma_0 + \delta)$. This rule is supplemented by results on wave amplitudes and shapes of crests.

The theory is applied (§§4 and 5) to Rossby waves excited in a beta-plane ocean by travelling patterns of wind stress. If a steady wind-stress pattern moves westward, semicircular waves of length $2\pi\sqrt{(U/\beta)}$ trail behind it, but signals are found also directly ahead, consisting of the disturbance integrated in the west–east direction and subjected to a 'low-pass filter' with respect to its north–south components of wave-number. An eastward-travelling pattern, by contrast, produces only a wake-like disturbance, calculated in detail in §4. The waves generated for intermediate directions of travel are identified, and the strong tendencies in all cases for westward intensification of transient currents are noted.

For example, a wind-stress pattern travelling 30° N. of E. leaves a trailing wedge of currents from W. to 30° S. of W. in the steady case. The influence on this conclusion of a finite duration T of such a pattern is investigated in §5 by Fourier analysis in time. The fate of Fourier components of frequency σ_0 depends on the ratio $L = \sigma_0/\surd(U\beta)$. If this is less than 1 for all σ_0 up to about $10/T$, then the disturbance retains its trailing character; on the other hand, any components with $L > 1$ have a much greater directional spread. Tidal terms make fairly small changes to the results, except that (in an ocean of depth H) they make the directional spread disappear for L greater than about $\surd(\beta gH/4f^2U)$.

Excitation of gravity waves in non-rotating fluid is briefly considered, including generation on deep water by a travelling oscillating disturbance (§6), and generation in a uniformly stratified fluid by a vertically moving obstacle (§7). The predicted wave shapes in the latter case, with cusps at a finite distance behind the obstacle, agree excellently (figures 7 and 8) with experiments by Mowbray (1966).

An exceptional case, in that part of $S(\sigma_0)$ is doubly covered, is generation by steady ($\sigma_0 = 0$) motion of an obstacle along the axis of uniformly rotating homogeneous fluid, the surface $S(0)$ being a sphere and two coincident planes. Whereas waves corresponding to points on the sphere trail behind the obstacle, the appropriate normals on the two planes point in opposite directions inside the sphere (§8), permitting the well-known formation of the 'Taylor column' ahead of the obstacle at low Rossby numbers.

Still more complicated, because fully three-dimensional, is the case when the obstacle moves at right angles to the axis of rotation (§9). At finite though small Rossby number it is impossible for the Taylor column formed near the body to extend to large distances from it, where on the contrary the disturbance is shown to take the form of slightly trailing cones, shown in cross-section in figure 12, containing waves whose crests have cusps on the boundaries of the cones. An estimate of the length of the Taylor column, as body dimension divided by Rossby number (for small enough kinematic viscosity), is made by considering the fit between the Taylor-column and wave-cone regions.

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1. Introduction

This paper is concerned with a homogeneous system whose undisturbed condition is stable, and in which small disturbances are possible, taking the form of plane waves satisfying an, in general, anisotropic dispersion relation. Within a region which is moving at constant velocity $\mathbf{U}$ through the system, a forcing process acts. The forcing function may be steady (independent of time), or oscillate with a fixed frequency, or be zero for times $t < 0$ and a prescribed function of time for $t > 0$. In all these cases the complex wave pattern generated by the forcing process is studied.

The general theory is given in §§2 and 3. The examples of its use which follow (§§4 to 9) are derived mainly from rotating fluid dynamics, forming a kind of continuation of the author's recent survey of that subject (Lighthill 1966, hereafter referred to as S). In rotating fluids, including the atmosphere and the oceans, many types of dispersive wave system are possible, and it is desirable to know how they can be excited by different forcing processes. Several possible forcing effects move relative to the fluid; for example, when an atmospheric disturbance travels over an ocean which it is perturbing, or when a disturbance that is fixed relative to the earth perturbs a wind blowing over it (in this case the disturbance is travelling relative to the air itself).

Previous discussions of currents generated either by steady (**S**, § 5) or transient (Longuet-Higgins 1965*b*) wind-stress distributions have taken their speed of travel over the ocean to be either zero or large compared with a typical group velocity. The discussion in §§4 and 5 below shows, however, that the intermediate case is of great importance. In fact the velocity of travel of the forcing effect in relation to its characteristic frequencies and wave-numbers appears to be of dominant significance in this problem, just as in the problem of the sound radiated by travelling eddies in a turbulent jet (Lighthill 1963).

Many problems involving forcing effects that are steady or of fixed frequency have been treated in the literature; for example the problem of § 8 (motion of an obstacle along the axis of rotating fluid). Difficulties have often been experienced, however, because the ensemble of solutions vanishing at infinity is a vast one, and *ad hoc* methods of selecting the solution satisfying the 'radiation condition' at infinity have been of very variable simplicity and effectiveness. This paper points out (§ 2) the extremely simple general rule, easy to infer from published papers but not so well known as it ought to be, for specifying the wave pattern that this solution involves.

A steady forcing effect may be so strong as to generate in its neighbourhood large disturbances not governed even approximately by the linear equations appropriate to small disturbances. The present theory can nevertheless be used to infer characteristics of the wave pattern set up far from the forcing region, where the disturbances are small enough for linear equations to apply. This is because the (admittedly unknown) non-linear terms in the equations which operate in the near field can be simply regarded as an additional forcing term whose region of application travels at the same speed.

For example, a large steady disturbance moving slowly through an extended body of uniformly rotating fluid at right angles to the axis can generate locally a 'Taylor column', but for non-zero Rossby number this cannot extend to infinity even in an inviscid fluid. In fact, the far disturbances must be small, and so take the form of inertial waves stationary with respect to the forcing disturbance. Their nature is worked out in § 9, where some suggestions are made also about how they match with the near-field 'Taylor-column' solution.

In addition, two problems in non-rotating fluids are considered. First, in § 6, as a link with the classical Kelvin ship-wave problem, the waves generated on deep water by an oscillatory disturbance travelling at speed U are studied. Results given by Eggers (1957) and others are confirmed, in opposition to incorrect predictions by Sretensky (1954), and subsumed within the general theory. It is shown that only oscillations of radian frequency less than $1{\cdot}62g/U$ can produce waves outside the normal ship-wave wedge. Secondly, in § 7, the shape of gravity waves excited in a uniformly stratified fluid by a vertically moving steady disturbance are calculated, in good agreement (figure 8) with experiments by Mowbray (1966).

2. General theory for steady or periodic forcing terms

Consider a system such that small disturbances to the undisturbed state are governed by a linear partial differential equation with independent variables x, y, z and t and constant coefficients, which we write

$$P\left(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z}\right)\phi = 0, \tag{1}$$

where P is a polynomial and ϕ is some variable specifying the disturbance. Then a plane wave

$$\phi = \phi_0 \exp\{i(-\sigma t + lx + my + nz)\} = \phi_0 \exp\{i(-\sigma t + \mathbf{k}.\mathbf{r})\} \tag{2}$$

can exist if the dispersion relation

$$P(\sigma, l, m, n) = 0 \tag{3}$$

is satisfied. On the other hand, because the undisturbed state is assumed stable, no solution of (3) exists with l, m, n real and the imaginary part of σ positive.

A forcing region is one where the right-hand side of (1) is replaced by a non-zero 'forcing term', which may represent the action of external forces on the system, and may also include substantial terms non-linear in the solution ϕ wherever disturbances are not small. In the forcing regions moving with uniform velocity $\mathbf{U}$ which this paper considers, steady forcing terms

$$f(\mathbf{r} - \mathbf{U}t), \tag{4}$$

where $\mathbf{r} = (x, y, z)$, are of particular interest. Sinusoidally varying forcing terms

$$e^{-i\sigma_0 t} f(\mathbf{r} - \mathbf{U}t) \tag{5}$$

are also of interest, and to save writing the theory will be given in this section in the more general case (5). The reader is asked, however, to bear in mind continually the specially important case $\sigma_0 = 0$.

We suppose that $f(\mathbf{r}) = f(x, y, z)$ vanishes outside a limited forcing region around the origin, and therefore can be written as a Fourier integral

$$f(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k}) e^{i\mathbf{k}.\mathbf{r}} dl\,dm\,dn, \tag{6}$$

where $F(\mathbf{k}) = F(l, m, n)$ is a regular function for all l, m, n. The equation

$$P\left(i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z}\right)\phi = e^{-i\sigma_0 t} f(\mathbf{r} - \mathbf{U}t), \tag{7}$$

with (6) used to rewrite the right-hand side, then has the formal solution

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F(\mathbf{k})\exp\{i[-\sigma_0 t + \mathbf{k}.(\mathbf{r} - \mathbf{U}t)]\}}{P(\sigma_0 + \mathbf{U}.\mathbf{k}, l, m, n)} dl\,dm\,dn. \tag{8}$$

This solution is not unique, however, in any system for which plane wave solutions satisfying (3) exist, because the denominator can then vanish for real l, m, n, in which case many determinations of the integral (8) are possible. However, only one is of physical significance, namely that obtained when the source strength has been built up to its present strength from zero, and the system has reached its steady state. This is obtained, for example, by replacing σ_0 by $\sigma_0 + i\epsilon$ (which gives an extra factor $e^{\epsilon t}$ in the forcing term (5) and also, because no σ with positive imaginary part satisfies (3), gives the integral (8) a determinate value) and then letting ϵ tend to zero. We shall see in § 3 that the same answer is obtained also if other modes of build-up of the forcing effect to its steady-state value are employed.

Lighthill (1965, § 7) has given the method for evaluating integrals such as (8) at distances from the forcing region large compared with its dimensions, using methods described earlier by Lighthill (1960). The asymptotic form of ϕ, as defined by replacing σ_0 by $\sigma_0 + i\epsilon$ in (8) and letting $\epsilon \to 0$, can be described as follows.

In wave-number $(l, m, n) = \mathbf{k}$ space, at each point of the surface

$$P(\sigma_0 + \mathbf{U}.\mathbf{k}, l, m, n) = 0 \tag{9}$$

on which the denominator of (8) vanishes, we draw an arrow normal to the surface, choosing from the two normal directions the one pointing towards the surface

$$P(\sigma_0 + \mathbf{U}.\mathbf{k} + \delta, l, m, n) = 0 \quad \text{with } \delta \text{ small and positive.} \tag{10}$$

In other words, the arrow is in the direction σ increasing. Then the waves (if any) found *in some particular direction* stretching out from the forcing region are those with $\mathbf{k} = (l, m, n)$ given by a *point* (if any) on the wave-number surface (9) *where the arrow is in that particular direction.* Their amplitude is asymptotically

$$\frac{4\pi^2}{|K|^{\frac{1}{2}}R}\,\frac{F(\mathbf{k})}{|\nabla P(\sigma_0 + \mathbf{U}.\mathbf{k}, l, m, n)|}, \tag{11}$$

where $R = |\mathbf{r} - \mathbf{U}t|$ means distance from the forcing region, ∇ is the operator grad with respect to $\mathbf{k} = (l, m, n)$ and K is the Gaussian curvature (product of principal curvatures) of the surface (9).

More strictly, ϕ is asymptotic to (11) provided $K \neq 0$, and falls off less rapidly than R^{-1} if $K = 0$. For example, we shall be concerned in what follows with cases of purely two-dimensional propagation, where there is no dependence on z at all. The wave-number surface is then cylindrical (so that $K = 0$); and its intersection with the plane $n = 0$, which we shall call the wave-number curve, alone determines the form of the waves generated. Equation (11) remains true with the first factor replaced by $(2\pi)^{\frac{3}{2}}/|\kappa|^{\frac{1}{2}}R^{\frac{1}{2}}$, where κ is the curvature of the wave-number curve. As a second example, a plane portion of the wave-number surface generates waves without attenuation, the first factor in (11) being replaced simply by 2π. Other examples are given by Lighthill (1960).

In a direction such that more than one point of the wave-number surface (9) has the arrow pointing in that direction, waves corresponding to each such point can be found superimposed on one another. The amplitude of each separately is determined by the above rules.

To explain physically the basic rule concerning the surface (9) and the arrows thereon, we note first that waves whose frequency is σ_0 *relative to* a forcing region moving at velocity $\mathbf{U}$ must have absolute frequency $\sigma_0 + \mathbf{U}.\mathbf{k}$ (the Doppler effect), and so their wave-number vector *must* lie on the surface (9). Furthermore, the group velocity for waves satisfying (3) is

$$\left(\frac{\partial \sigma}{\partial l}, \frac{\partial \sigma}{\partial m}, \frac{\partial \sigma}{\partial n}\right) = -\frac{\nabla P}{\partial P/\partial \sigma}. \tag{12}$$

Now, at time $t = 0$ when the forcing region is around the origin, the position of a wave group created earlier, at time $t = -T$ when it was around the point $-\mathbf{U}T$, and propagating since then at the group velocity, must be

$$-\mathbf{U}T - \frac{\nabla P}{\partial P/\partial \sigma} T, \tag{13}$$

which does indeed lie in the direction of the arrow defined above.

Lighthill (1965), following Whitham (1960), gives also an interpretation of the amplitude variation (11). However, the only feature of this used below is the fairly obvious one, that waves will be generated corresponding only to those parts of the wave-number surface (9) for which the Fourier transform $F(\mathbf{k})$ of the forcing term is not negligibly small.

The shapes of wave-crests and other surfaces of constant phase can be deduced from the above rules (Lighthill 1960). Each is in fact the 'reciprocal polar' of the wave-number surface (9), that is, the locus of the poles of its tangent planes with respect to the origin. Analytically, it is the locus of the points

$$A\frac{\nabla P(\sigma_0 + \mathbf{U}.\mathbf{k}, \mathbf{k})}{\mathbf{k}.\nabla P(\sigma_0 + \mathbf{U}.\mathbf{k}, \mathbf{k})}, \tag{14}$$

where A is a constant.

In the special case $\sigma_0 = 0$ the surface (9) becomes

$$P(\mathbf{U}.\mathbf{k}, l, m, n) = 0, \tag{15}$$

which may be interpreted as a statement that a steady forcing effect can generate only waves whose crests are stationary relative to the velocity of the forcing region. This physically plausible idea (which can also be expressed by saying that the component of the forcing region's velocity in the direction of the phase velocity is exactly equal to the phase velocity) is proved to be of general validity by the present mathematical arguments.

3. General theory for transient forcing terms

Before giving examples of the results for steady or periodic forcing terms, we shall briefly obtain the corresponding results for a transient type of forcing term $f(\mathbf{r} - \mathbf{U}t, t)$, where $f(\mathbf{r}, t)$ is zero for $t < 0$, and also, as in §2, is assumed zero outside a limited region of space around the origin. Under these circumstances it can be written as a Fourier integral

$$f(\mathbf{r}, t) = \int_{i\epsilon - \infty}^{i\epsilon + \infty} e^{-i\sigma t} d\sigma \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\sigma, \mathbf{k}) e^{i\mathbf{k}.\mathbf{r}} dl\, dm\, dn, \tag{16}$$

where $\epsilon > 0$ and $F(\sigma, \mathbf{k})$ has no singularities where the imaginary part of σ is positive.

The formal solution corresponding to (8) when the right-hand side of (7) is replaced by $f(\mathbf{r}-\mathbf{U}t, t)$ is

$$\phi = \int_{i\epsilon-\infty}^{i\epsilon+\infty} e^{-i\sigma t} d\sigma \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F(\sigma, \mathbf{k}) \exp[i\mathbf{k}.(\mathbf{r}-\mathbf{U}t)]\, dl\, dm\, dn}{P(\sigma+\mathbf{U}.\mathbf{k}, l, m, n)}. \tag{17}$$

The asymptotic behaviour of (17) at large distances from the source will here be considered, first for a forcing term of finite duration and then in the case of one which becomes purely oscillatory after a finite time.

Lighthill (1965, §6) gave the solution for a source of finite duration, using the ideas of Lighthill (1960, appendix B) in a simplified form. The results will here be quoted in terms of the forms of the surfaces (9) for different values of the frequency σ_0.

The waves (if any) of frequency σ_0 found in some particular direction stretching out from the forcing region are those with $\mathbf{k} = (l, m, n)$ given by a point (if any) on the wave-number surface (9) for which the arrow is in that direction. Their amplitude is proportional to

$$\frac{F(\sigma_0, l, m, n)}{R^{\frac{3}{2}}}, \tag{18}$$

where $R = |\mathbf{r}-\mathbf{U}t|$ as before, and where the explicit form of the factor of proportionality depends like that in (11) on the geometry of the surfaces (9) but will not here be required. The energy density, which is proportional to the square of (18), falls off like R^{-3} instead of like R^{-2} because dispersion makes a transient disturbance grow outwards as a wave group filling a region whose volume expands in all three of its dimensions. The position at which the waves are found at time t is given by t times the appropriate group velocity (12).

This asymptotic result for the transient disturbance is derived from the singularities of the integrand in (17), which for $\sigma = \sigma_0$ are on the surface (9). On the other hand, for a source whose action continues indefinitely, additional singularities can arise in the integrand, due to singularities in F itself. For example, if

$$f(\mathbf{r}, t) = e^{-i\sigma_0 t} f(\mathbf{r}) \tag{19}$$

meaning that the simple harmonic forcing term (5) is 'switched on' at time $t = 0$, then

$$F(\sigma, \mathbf{k}) = \frac{F(\mathbf{k})}{2\pi i(\sigma-\sigma_0)}, \tag{20}$$

and this possesses a singularity at $\sigma = \sigma_0$. The asymptotic behaviour of (17) is then best obtained by first using the method of §2 for the inner integral (in which, as in §2, the imaginary part of σ is $+\epsilon$), after which integration with respect to σ gives exactly the result of §2, because division by $2\pi i(\sigma-\sigma_0)$ followed by integration replaces σ by σ_0.

This means that we obtain asymptotically the same steady-state solution, by suddenly switching on the source term (5) and waiting, as we did in §2 by allowing the strength to grow gradually from zero like $e^{\epsilon t}$. A much more general forcing term is the sum of (19) and an arbitrary source term of finite duration. This represents, indeed, a completely general forcing term starting from zero at time

$t = 0$ and becoming sinusoidal after a finite time. The solution in this case is the sum of the R^{-1} term of §2 and the $R^{-\frac{4}{3}}$ term of (18). Ultimately the former dominates, and we once again receive the solution which satisfies the radiation condition as the limiting result after the source has for a long time assumed the sinusoidal form.

Summary of sections 2 *and* 3

Sections 2 and 3 can be summarized by saying that the waves generated by forcing effects moving at velocity $\mathbf{U}$ are determined above all by the shapes of the surfaces $S(\sigma_0)$ in wave-number space given by equation (9). Arrows normal to $S(\sigma_0)$ pointing in the direction of $S(\sigma_0+\delta)$ indicate in what direction stretching out from the source region waves of given frequency σ_0 and wave-number (l, m, n) will be found. But only those parts of $S(\sigma_0)$ where the forcing term's Fourier transform ($F(\mathbf{k})$ for a steady disturbance or $F(\sigma, \mathbf{k})$ for a transient disturbance) takes significant values will produce significant waves. In directions corresponding to those parts, the surfaces of constant phase for an oscillating disturbance of frequency σ_0 have the shape of the reciprocal polar of the surface $S(\sigma_0)$.

4. Rossby waves excited by a travelling steady disturbance

The method of this paper will first be applied to a travelling steady forcing effect generating Rossby waves in a 'beta-plane ocean'. Studies by Longuet-Higgins (1964, 1965*a*) appear to indicate that waves in an ocean of uniform depth at frequencies large compared with the Coriolis parameter can be approximated reasonably well by divergenceless Rossby waves on a beta-plane, although a still better approximation for the lower wave-numbers is obtained by including a tidal term in the dispersion equation (see also **S**, (46)); the effect of this term on the results will be noted at the end of the present section.

Divergenceless Rossby waves on a beta-plane (**S**, (36)) satisfy

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\beta\frac{\partial\psi}{\partial x} = 0, \tag{21}$$

where the x-direction is eastward and β is the gradient of Coriolis parameter in the northward y-direction. Multiplying (21) by $-i$ (for convenience) before comparison with (1), we obtain for this two-dimensional system

$$P(\sigma, l, m) = \sigma(l^2+m^2)+\beta l. \tag{22}$$

A classic problem is the generation of Rossby waves by a steady westward-moving forcing effect. Such a means of excitement, relatively rare in the ocean, corresponds in the atmosphere to a commoner situation, generation by a steady eastward-moving wind blowing past a topographical feature.

If the forcing effect moves with velocity $(-U, 0)$, then the wave-number curve $S(0)$, given by equation (9) with $\sigma_0 = 0$, is

$$-Ul(l^2+m^2)+\beta l = 0, \tag{23}$$

which consists of the straight line $l = 0$ and the circle

$$l^2+m^2 = \beta/U. \tag{24}$$

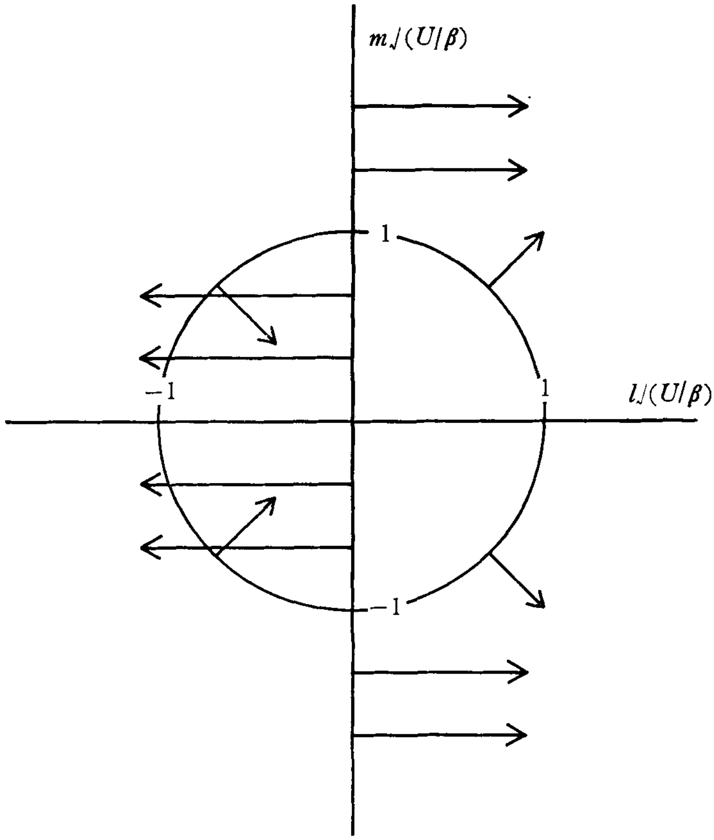

FIGURE 1. Wave-number curve for Rossby waves generated on a beta-plane ocean by a steady forcing effect travelling westward, with velocity $(-U, 0)$.

The literature has laid particular emphasis on the waves (24), of uniform length $2\pi\sqrt{(U/\beta)}$ and arbitrary direction.

But according to §2 it is essential to study not only the wave-number curve $S(0)$, here given by (23), but also the arrows normal to it pointing towards $S(+\delta)$, the curve defined by $P(-Ul+\delta, l, m) = 0$. Now it is easy to show that the change in l (say) for fixed m in going from $S(0)$ to $S(\delta)$ for small δ is asymptotically $(l^2+m^2)\delta/[U(3l^2+m^2)-\beta]$, and hence that the required arrows are as in figure 1. This means that the waves satisfying (24), with circular wave-crests, trail to the east of the westward-moving disturbance, filling the eastward-facing hemisphere behind it. The physical explanation of this was given in **S**, §7.

In addition, disturbances with l and σ zero, that is, independent of x and t, are possible, with those whose meridional wave-number $|m|$ exceeds $\sqrt{(\beta/U)}$ appearing behind the forcing region (that is, to the east), and those for which it is less appearing in front (to the west). Physically, this results from the rule due to Longuet-Higgins (1964) that the group velocity of Rossby waves makes an angle with the eastward direction twice that which the wave-number vector $\mathbf{k}$ makes, and is of magnitude β/k^2. For waves with $l = 0$, that is, with east–west crests, the group velocity β/m^2 is westward (although the phase velocity is zero), and exceeds U if and only if

$$|m| < (\beta/U)^{\frac{1}{2}}. \tag{25}$$

These waves propagate (on the dissipationless model here used) without attenuation, because the associated part of the wave-number curve is a straight line. The disturbance that extends ahead (westward) of the obstacle is then the transverse disturbance created at the obstacle† modified by a 'low-pass filter' passing

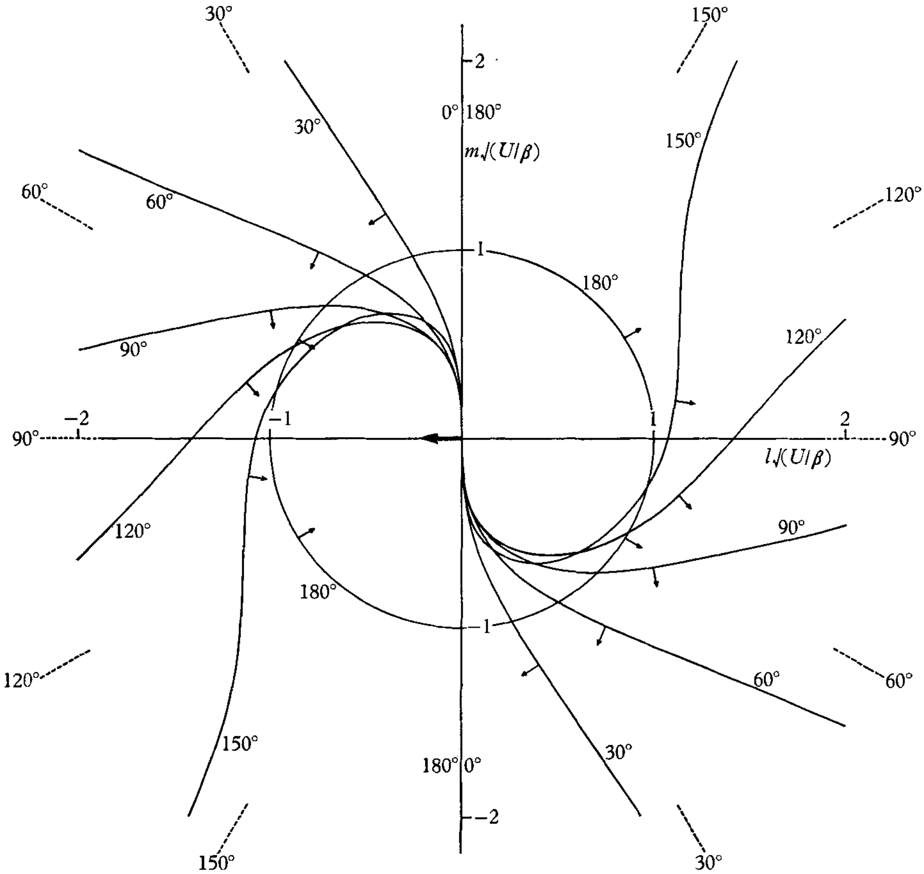

FIGURE 2. Wave-number curves for Rossby waves generated on a beta-plane ocean by a steady forcing effect travelling with uniform velocity U in directions making positive angles α = 0°, 30°, 60°, 90°, 120°, 150° and 180° (marked on the curves) with the eastward direction. Arrows are omitted on the m-axis, which both is the whole curve α = 0° (arrows westward only) and also is part of the curve α = 180° (arrows as in figure 1). - - - - - -, asymptotes.

only wave-numbers below $\surd(\beta/U)$. The disturbance extending to the east of the obstacle has been subjected to the complementary high-pass filter.

The situation with an eastward-moving steady forcing effect is much simpler. The sign of the first term in (23) being changed, the wave-number curve is merely the axis $l = 0$, and all arrows point to the west. There is therefore merely a long

† Strictly speaking, after integration in the east–west direction, because the $F(\mathbf{k})$ term in (11) for $l = 0$ represents the integral of $f(\mathbf{r})$ with respect to x from $-\infty$ to ∞.

straight unattenuated disturbance trailing behind the forcing region to the west of it, exactly as found in the experiments of Fultz & Long (1951).

Wave-number curves for steady forcing effects moving at an angle α measured in the positive sense from the eastward direction are given in figure 2. They satisfy

$$U(l\cos\alpha+m\sin\alpha)(l^2+m^2)+\beta l = 0. \tag{26}$$

The point of inflexion of each curve at the origin, where the arrow points westward in every case, means that, whenever the forcing term has significant wave-number components in this region, a strong signal will be found to the west of the disturbance.† (It may be noted, for example, that amplitude attenuation of two-dimensional waves from such a point of inflexion is like $R^{-\frac{1}{3}}$ instead of $R^{-\frac{1}{2}}$.)

Steady travelling forcing effects in general, then, excite disturbances to the west. Disturbances to the east will be excited only if the velocity of travel has a substantial westward component. This is rare in oceans in the temperate zones, which partly explains why fluctuations of current in the North Atlantic have often been observed to be much greater to the west than to the east (Swallow 1961).

It is of interest to calculate quantitatively the ocean movements in one particularly simple but relevant case, a steady depression crossing the ocean from west to east. If the wind-stress per unit mass of water has east and north components $X(x-Ut,y)$ and $Y(x-Ut,y)$ respectively, then the rate of change of vertical vorticity due to wind stress is the wind-stress curl

$$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y} = c(x-Ut,y), \tag{27}$$

and so the forcing term which must be added to the right-hand side of (21) is $-c(x-Ut,y)$ (minus because the vorticity is $-\nabla^2\psi$). If $C(l,m)$ is the Fourier transform of $c(x,y)$, defined as in (6) but in two dimensions, then the formal solution for ψ corresponding to (8) is

$$\psi = i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{C(l,m)\exp\{i[l(x-Ut)+my]\}}{Ul(l^2+m^2)+\beta l}\,dl\,dm. \tag{28}$$

We saw above that disturbances far from the forcing region are substantial only due west of it, where $x-Ut$ is large and negative but y is not large. Estimation, either by the rule for plane portions of the wave-number surface given after equation (11), or by direct asymptotic calculation of (28) (using the determination obtained by replacing Ul by $Ul+i\epsilon$ and letting $\epsilon\to 0$) gives that, as $x-Ut\to -\infty$,

$$\psi \sim 2\pi\int_{-\infty}^{\infty}\frac{C(0,m)\,e^{imy}}{Um^2+\beta}\,dm \tag{29}$$

$$= \frac{1}{2\sqrt{(\beta U)}}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}c(x_1,y_1)\exp\left[-|y_1-y|\sqrt{(\beta/U)}\right]dy_1, \tag{30}$$

where to obtain (30) from (29) Parseval's theorem has been used.

† Figure 2 shows how, as the wave-number increases relative to $\sqrt{(\beta/U)}$, the direction in which the waves are found, measured in the positive sense from the westward direction, increases from 0 to a maximum just greater than α before falling to its final asympotic value α.

Two limiting forms of (30) are of interest. First, when U, the velocity of convection of the forcing region, supposed of dimension L, is small compared with βL^2, then the form (30) of ψ as $x-Ut \to -\infty$ becomes approximately

$$\psi = \frac{1}{\beta}\int_{-\infty}^{\infty} c(x_1, y)\, dx_1. \tag{31}$$

This agrees with the solution of Sverdrup's classical steady-flow problem, that is,

$$\beta(\partial\psi/\partial x) = -c(x,y), \tag{32}$$

provided that ψ be taken zero to the east of the disturbance. Often elaborate boundary-layer arguments have been used to justify this boundary condition on ψ (S, §5), but the present work shows it as an immediate consequence of proper application of the radiation condition.

Secondly, when U is large compared with βL^2, equation (30) differentiated with respect to y gives approximately

$$u = \frac{\partial\psi}{\partial y} \doteqdot \frac{1}{2U}\int_{-\infty}^{\infty} dx_1 \left(\int_{y}^{\infty} c(x_1, y_1)\, dy_1 - \int_{-\infty}^{y} c(x_1, y_1)\, dy_1\right), \tag{33}$$

which with expression (27) for c means simply

$$u = \frac{1}{U}\int_{-\infty}^{\infty} X(x_1, y)\, dx_1, \tag{34}$$

stating that water, uninfluenced by the beta-effect, has been accelerated directly by the force $X(x-Ut, y)$ per unit mass as the forcing region passes it. Water south of the centre of the depression is dragged eastward behind it, and water north of the centre is pushed westward. In this case, the group velocity β/k^2 of disturbance is small compared with U, so that only the water which the disturbance has actually passed over can be affected.

By contrast, in the case $U \ll \beta L^2$, the result (31), which in terms of u and X can be written

$$u = -\frac{1}{\beta}\frac{d^2}{dy^2}\int_{-\infty}^{\infty} X(x_1, y)\, dx_1, \tag{35}$$

will hold even far to the west of where the depression may have originated, since the group velocity is much greater than U. When expression (35) is valid, it is smaller than (34), and vice versa; whereas when U and βL^2 are of the same order, the expression

$$u = \frac{1}{U}\int_{-\infty}^{\infty} dx_1 \left[X(x_1, y) - \frac{1}{2}\left(\frac{\beta}{U}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} X(x_1, y_1) \exp\{-|y_1 - y|\sqrt{(\beta/U)}\}\, dy_1\right], \tag{36}$$

obtainable from (30) by differentiating with respect to y and then integrating by parts, shows that u again normally falls below the limiting value (34).

Probably the most interesting conclusion from this section is that, if the problem of steady wind-driven ocean currents is regarded as a limiting case of currents driven by a travelling forcing effect as the speed of travel tends to zero, then Sverdrup's solution with no disturbance to the east is obtained. (The conclusion is unaltered for a westward-moving forcing region, because the limitation (25) to wave-numbers less than $\surd(\beta/U)$ ceases to be restrictive as $U \to 0$.) This boundary condition is appropriate, therefore, for more fundamental reasons than have usually been given; physically, because group velocity is westward for north–south wave-numbers.

The results of this section are not much changed when the tidal term f^2/gH for an ocean of constant depth H is added (**S**, (46)) to l^2+m^2 in (22), (23) and (24). The critical wave-number for a westward-moving forcing effect becomes

$$\left(\frac{\beta}{U}-\frac{f^2}{gH}\right)^{\frac{1}{2}}. \tag{37}$$

The curves in figure 2 are slightly modified, to pass through the origin at a small positive angle

$$\tan^{-1}\left[\frac{\sin\alpha}{(\beta gH/f^2U)+\cos\alpha}\right] \tag{38}$$

to the m-axis. Accordingly, the low wave-number disturbances are to be found at this small positive angle to the westward direction. These changes are not really important provided that U is small compared with

$$\frac{\beta gH}{f^2} = (11\,\mathrm{m/s})\left(\frac{\cos\theta}{\sin^2\theta}\right)(H\,\mathrm{in\,km}), \tag{39}$$

which is likely to be the case except perhaps at rather high latitudes θ, or small depths H.

5. Rossby waves excited by a travelling transient disturbance

In this section the generation of Rossby waves in a beta-plane ocean by travelling forcing effects of transient character is studied. As in §4, the effect of a tidal term is considered only at the end of the section.

Longuet-Higgins (1965*b*) gave an excellent account of transient currents generated (i) by a stationary transient forcing effect, and (ii) by a transient forcing effect that moves 'very rapidly', that is, much faster than the group velocity of the waves produced. The present solution is valid not only in these two extreme cases, but also for those intermediate speeds of travel which are often important in practice. It is complementary also in another way to the work of Longuet-Higgins (1965*b*), which is concerned with instantaneous impulse-type (delta-function) forcing effects, so that waves of all frequencies, however high, can be produced. Here we consider disturbances of non-zero duration, which normally will not excite waves of very high frequency. We shall see that their exclusion makes significant qualitative differences to the conclusions.

The wind-stress curl, then, has the form $f(\mathbf{r}-\mathbf{U}t,t)$ as in §3, where the function $f(\mathbf{r},t)$ vanishes except in a finite region of $\mathbf{r}$ and within a finite time interval, and can be expressed as a Fourier integral as in (16). We shall assume such smooth variation of $f(\mathbf{r},t)$ that its Fourier transform $F(\sigma,k)$ is small for frequencies σ exceeding a frequency σ_1 characteristic of the disturbance, or for wave-numbers k exceeding a characteristic wave-number k_1. Here σ_1 might be about 10 divided by the 'duration' T of the disturbance, since, for example, a Gaussian transient proportional to $\exp[-10(t/T)^2]$, with amplitude 8 % of its maximum at $t = \pm\frac{1}{2}T$, has Fourier transform less than 8 % of *its* maximum for $\sigma > 10/T$.

For each frequency σ_0 less than σ_1, the waves generated are those specified by the curve $S(\sigma_0)$ in the wave-number plane. This curve, which if the direction of travel makes a positive angle α with the eastward direction has the equation

$$[\sigma_0 + U(l\cos\alpha + m\sin\alpha)](l^2+m^2) + \beta l = 0 \tag{40}$$

by (9) and (22), is drawn in figure 3 for various σ_0 when $\alpha = 30°$. This particular case was chosen because it appeared in §4 that steady forcing effects when α is relatively small can generate Rossby waves only in a limited sector trailing

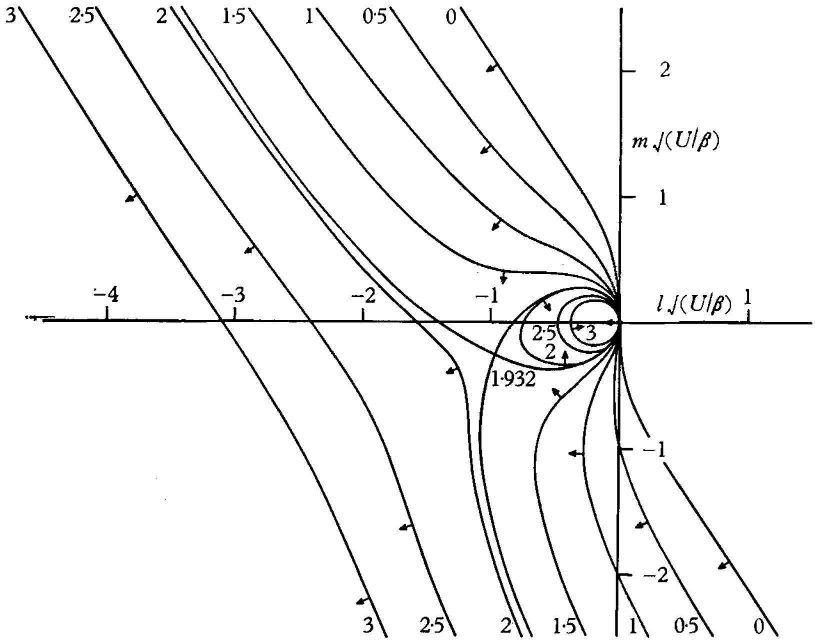

FIGURE 3. Wave-number curves for Rossby waves generated on a beta-plane ocean by an oscillatory forcing effect travelling with uniform velocity U in a direction making a positive angle 30° with the eastward direction. The number marked on each curve is the value of $L = \sigma_0/\surd(U\beta)$, where σ_0 is the frequency.

behind the disturbance, and it is desirable to find out (without limitation to the over-special case $\alpha = 0$) whether this conclusion remains valid for transient forcing effects travelling in directions typical of temperate-zone conditions.

It is seen that the shape of $S(\sigma_0)$ depends critically on the value of a frequency parameter

$$\sigma_0/\surd(U\beta) = L, \tag{41}$$

say. It is close to the form $S(0)$ used in §4 to investigate steady forcing effects only when L is small. Big changes of form occur for values of L around unity, and for

$$L > 2\cos\tfrac{1}{2}\alpha \tag{42}$$

(here $L > 1{\cdot}932$) the curve splits into two. For larger values of L, the two parts approximate closer and closer to a straight line and a circle. For a non-travelling forcing effect ($U \to 0$, $L \to \infty$) the circle alone is left (as would be expected from the work of Longuet-Higgins 1965*b*).

When $\sigma_0 = \sigma_1$, the highest frequency characteristic of the forcing effect, values of L satisfying (42) may be found for disturbances of relatively short duration. For example, a forcing effect of duration 4 days travelling at 10 m/s at latitude 45° has $L = 2{\cdot}5$ for $\sigma_0 = \sigma_1$. However, all frequencies below σ_1 will in general be significant, and an important zero-frequency component is in particular present if the time integral of the disturbance is non-zero. All the values of L in figure 3 are likely to be found together, therefore.

Figure 3 shows that for the lower values of L, around 0 to 1·0, the currents generated trail once more in a narrow wedge behind the forcing region, but that for the higher values of L Rossby waves all round it may be generated. However, only waves of very small wave-number $k = \sqrt{(l^2+m^2)}$, say with $k\sqrt{(U/\beta)}$ less than about 1, will be found all round it, physically for the same reason that led to the condition (25). For the example just quoted, this would limit such waves to those of length exceeding 5000 km. For a different example, with a forcing effect of duration 14 days travelling at 4 m/s at latitude 45°, the maximum value of L would be 1 and all disturbances would be trailing.

A still more pronounced tendency for the wave pattern to trail exclusively behind the disturbance is found when gravity effects are taken into account, as at the end of § 3, by adding a term f^2/gH in equation (22). This term must be added to l^2+m^2 also in (40), which modifies the curves in figure 3 mainly *near the origin*, where l^2+m^2 is small. The modification to the curve $L = 0$, already noted in § 4, is that it passes through the origin at the small positive angle (38) to the m-axis. The other curves, however, cease to pass through the origin, where they are displaced, in fact, towards the left.

For the larger values of L, the new term actually reduces the size of the nearby circular branch of the curve, somewhat as found by Longuet-Higgins (1965*a*, *b*) with U neglected. It approximates then to the circle

$$\sigma_0\left(l^2+m^2+\frac{f^2}{gH}\right)+\beta l = 0, \tag{43}$$

with radius

$$\left(\frac{\beta^2}{4\sigma_0^2}-\frac{f^2}{gH}\right)^{\frac{1}{2}}, \tag{44}$$

and vanishes altogether when σ_0 exceeds

$$\frac{\beta\sqrt{(gH)}}{2f} = (0{\cdot}7\cot\theta)\,(\text{depth in km})^{\frac{1}{2}}\,(\text{days})^{-1}, \tag{45}$$

or when L exceeds $(\beta gH/4f^2U)^{\frac{1}{2}}$. In the first example quoted above, the values of L for which the waves were not trailing, namely 1 to 2·5, *all* correspond to values of σ_0 which already exceed the limit (45) for depths less than 2 km; and, even for a depth of 4 km, only a minimal amount of wave energy, with L between 1 and 1·5, could be found ahead of the forcing region.

6. Surface gravity waves generated by a travelling oscillating disturbance

Among the best-known wave combinations due to travelling forcing effects is Kelvin's pattern of surface gravity waves, set up by a ship in steady motion, and shown by him to be confined within a wedge of semi-angle $19\frac{1}{2}°$. The method of this paper is now used to study surface gravity waves generated by a travelling disturbance that is not steady but oscillatory with frequency σ_0, so that Kelvin's ship waves are the special case $\sigma_0 = 0$.

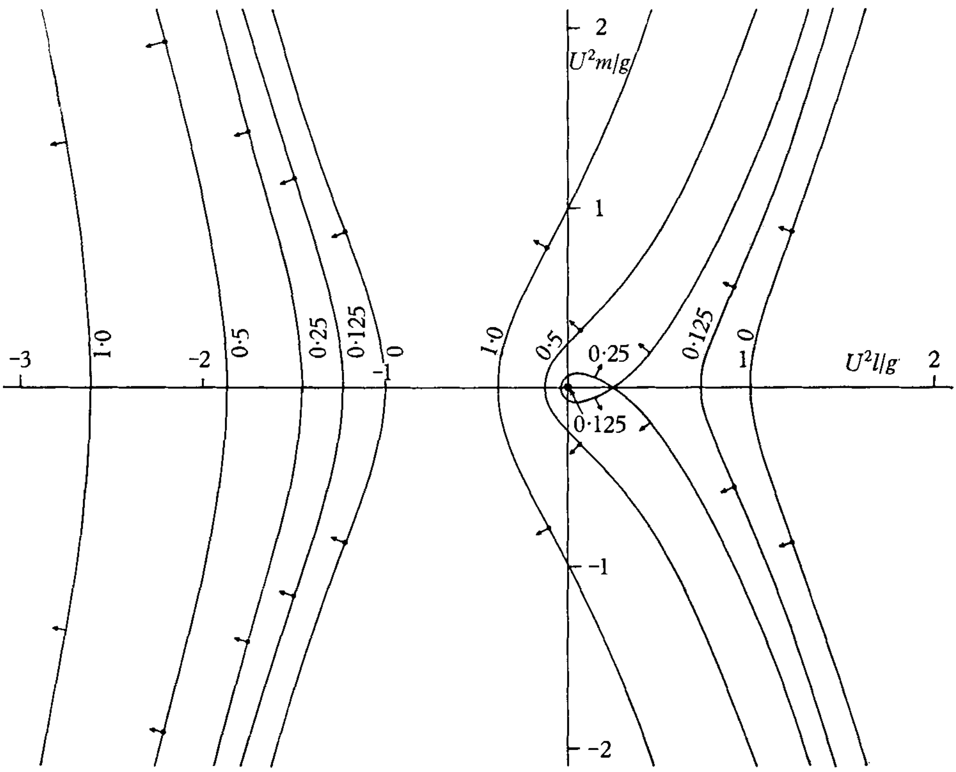

FIGURE 4. Wave-number curves for surface gravity waves, generated by an oscillatory forcing effect with various frequencies σ_0, travelling with velocity $(U, 0)$ over deep water. The numbers 0, 0·125, 0·25, 0·5 and 1·0 on the curves give the values of $U\sigma_0/g$ in each case. On certain branches of the curves, which correspond to a wedge of waves, points of inflexion (corresponding to waves on the boundary of the wedge) are marked by a spot.

This problem, like some others studied in this paper, has been treated by various writers, with very different results. Our object here is to show that the correct placing of the waves follows immediately from the general theory of §2, which, in fact, supports the work of Eggers (1957) and Newman (1959) and others against that of Sretensky (1954).

For this two-dimensional system the dispersion relation takes the form (3), where

$$P(\sigma, l, m) = \sigma^4 - g^2(l^2 + m^2). \tag{46}$$

Hence equation (9) for the surface $S(\sigma_0)$ becomes

$$(\sigma_0 + Ul)^4 = g^2(l^2 + m^2), \tag{47}$$

where the forcing effect has frequency σ_0 and travels with velocity $(U, 0)$.

The surface $S(\sigma_0)$ is shown in figure 4 for various values of the ratio $U\sigma_0/g$. Kelvin's ship waves have wave-numbers on $S(0)$, and fill a backward-trailing wedge of semi-angle $19\frac{1}{2}°$ because the arrows on $S(0)$ (pointing towards $S(+\delta)$) do so. When $U\sigma_0/g$ takes small positive values (for example, 0·125) the two sheets of which $S(0)$ is composed are in $S(\sigma_0)$ both displaced to the left. Certain waves with larger wave-numbers, associated with the left-hand sheet, fill a narrower wedge than before, and other waves with smaller wave-numbers, associated with

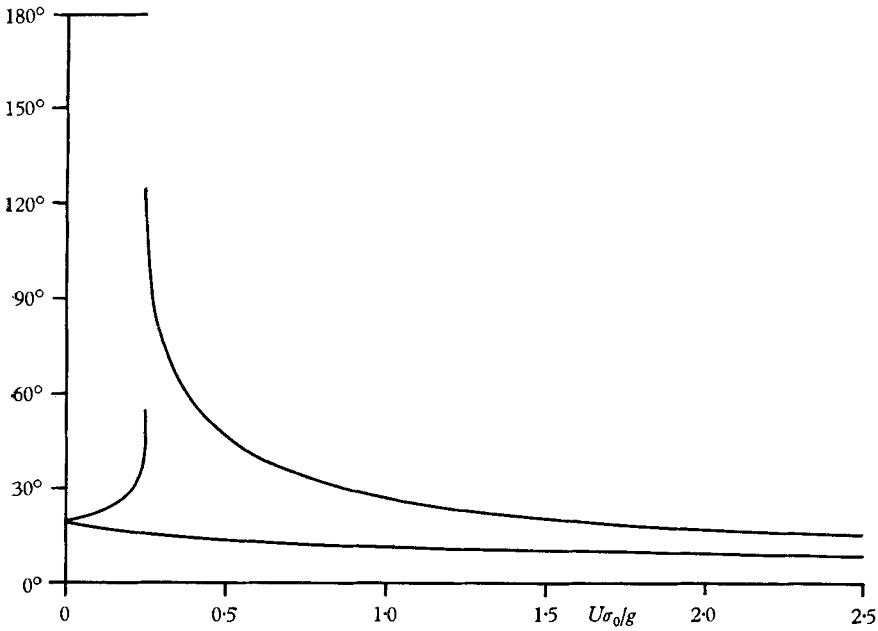

FIGURE 5. Illustrating how the semi-angles of the wedges within which the waves represented in figure 4 lie change with $U\sigma_0/g$.

the right-hand sheet, fill a wider wedge. (The point of inflexion, marked on each curve, corresponds to waves on the boundary of such a wedge.) $S(\sigma_0)$ includes, in addition, a small oval region involving very small wave-numbers (of the order of σ_0^2/g), and, if the forcing effect has components with such wave-numbers, then the associated waves are disposed in all directions around it.

Transition to a new régime occurs at $U\sigma_0/g = 0{\cdot}25$ (where $S(\sigma_0)$ crosses itself), beyond which there are only two branches of $S(\sigma_0)$, and the waves associated with each lie within a certain wedge. Figure 5 plots the semi-angles of the wedges within which the waves associated with the left-hand and right-hand branches lie as a function of $U\sigma_0/g$. (For $U\sigma_0/g < 0{\cdot}25$, a value of 180° is also included, to represent the fact that waves associated with the small oval branch are found in all directions around the forcing region.)

One interesting conclusion is that waves are found outside the Kelvin wedge, associated with the steady part of any composite travelling disturbance, only for frequencies σ_0 satisfying $U\sigma_0/g < 1{\cdot}63$. Another is that waves are found in front of the obstacle (that is, in the forward-facing semicircle) only if $U\sigma_0/g < 0{\cdot}27$.

7. Internal gravity waves generated by a vertically moving steady disturbance

Gravity waves in a uniformly stratified medium are next discussed. Generation by a vertically moving steady disturbance is of particular interest owing to the importance in certain stably stratified regions of the atmosphere of the phenomenon known as a 'thermal', that is, a rising localized region of hot air. The question of what gravity waves, if any, a 'thermal' generates may be treated

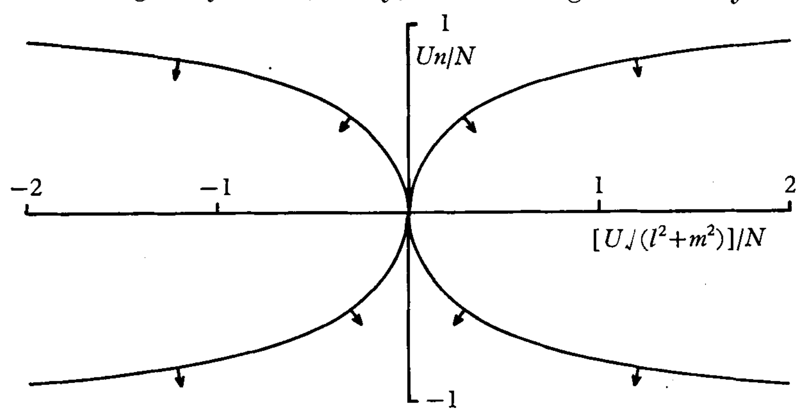

FIGURE 6. Wave-number surface for internal gravity waves generated by steady vertical motion of an obstacle with velocity (0, 0, U) through a uniformly stratified medium with Väisälä–Brunt frequency N.

approximately by regarding it as a travelling steady disturbance (see Warren (1960), who gave an excellent analysis of wave-making resistance on this assumption). Another reason for interest in this case is that experiments suitable for detailed comparison with theory have been made by Mowbray (1966), using a tank of uniformly stratified salt solution.

With the z-axis vertical, the vertical component of velocity w in gravity waves satisfies

$$\frac{\partial^2}{\partial t^2}\nabla^2 w = N^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right), \tag{48}$$

where N is the Väisälä–Brunt frequency. Additional terms significant only if the wavelength is comparable with the scale height have been omitted from (48), and in what follows a uniformly stratified medium with constant N will alone be considered. Accordingly, in this three-dimensional problem,

$$P(\sigma, l, m, n) = \sigma^2(l^2+m^2+n^2)-N^2(l^2+m^2), \tag{49}$$

and for steady forcing effects travelling with velocity $(0, 0, U)$ equation (9) with $\sigma_0 = 0$ for the surface of revolution $S(0)$ becomes

$$U^2n^2(l^2+m^2+n^2) = N^2(l^2+m^2). \tag{50}$$

Figure 6 gives the surface (50) in meridian section, with the arrows aimed towards $S(+\delta)$. This is a problem where all waves trail behind the forcing region. They are found, in fact, only *below* a rising obstacle, or *above* a falling one. Shorter waves, with wave-number comparable with N/U or greater, are found close to the path of the obstacle, but longer waves, with wave-number small compared with N/U, are found (if the forcing effect includes components with those wave-numbers) at places whose radius vector from the forcing region makes all angles up to 90° with the path.

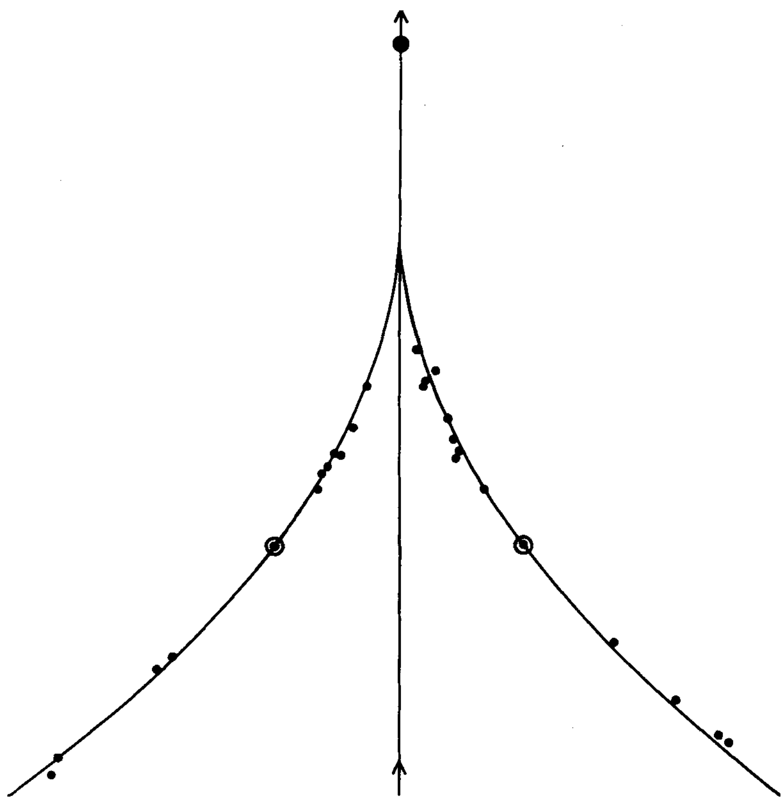

FIGURE 8. Shape of a surface of constant phase for internal gravity waves generated by steady vertical motion of a sphere (shown at top of diagram) through a uniformly stratified medium. Shape is normalized so that the points on the lines through the obstacle making an angle $\tan^{-1}(\frac{1}{4})$ with the vertical are in the ringed positions. Curve: theoretical shape. Points: experimental results from figure 7 and from another similar photograph (Mowbray 1966).

This is a case where the shapes of wave-crests (surfaces of constant phase) are well worth calculating, for comparison with Mowbray's experimental observations. Figure 7 is a schlieren photograph of a sphere being raised at uniform velocity through uniformly stratified salt solution. The loci of maximum darkness are surfaces of constant phase (in fact, nodal surfaces with respect to density). The shape of several of these was measured and plotted on a single diagram (figure 8) after being scaled so that the points at an angle $\tan^{-1}(\frac{1}{4})$ behind the centre of the sphere (ringed in figure 8) are the same for each.

The curve in figure 8 is the theoretical surface of constant phase, given by equation (15) as

$$A\left(\frac{(U^2n^2-N^2)\,l,\,(U^2n^2-N^2)\,m,\,U^2(l^2+m^2+2n^2)\,n}{N^2(l^2+m^2)}\right), \tag{51}$$

where l, m, n satisfy (50). It was plotted in terms of the parameter Un/N as

$$\frac{N\sqrt{(x^2+y^2)}}{AU}=\frac{(1-Un/N)^{\frac{3}{2}}}{(Un/N)^2},\quad \frac{Nz}{AU}=\frac{2}{Un/N}-(Un/N). \tag{52}$$

The agreement with the experimental points is seen to be good.

8. Motion of obstacle along axis of rotating fluid

Various special geometrical features of the surface $S(\sigma_0)$, as discussed in §2 above, and in more detail by Lighthill (1960), can call for special treatment. One not there mentioned, and yet arising in a problem that has been studied at great length in the literature of rotating fluids, as well as being of independent interest, is the case when part of $S(\sigma_0)$ is twice covered; that is, when the surface contains two coincident portions. The difficulties experienced by many writers on the subject now to be described can be partly related to this rather unusual circumstance.

Through a large body of homogeneous fluid, in uniform rotation with angular velocity Ω, an obstacle moves with constant velocity U along the axis of rotation, which is the z-axis. As explained in §2, the motion at large distances from the obstacle would be expected to constitute a small perturbation of the state of uniform rotation, and therefore (**S**, §6) to take the form of inertial waves, in which all components of the fluid velocity $\mathbf{v}$ (in a rotating frame of reference) satisfy

$$\frac{\partial^2}{\partial t^2}\nabla^2\mathbf{v}+4\Omega^2\frac{\partial^2\mathbf{v}}{\partial z^2}=0. \tag{53}$$

Comparison with (1) shows that

$$P(\sigma,l,m,n)=\sigma^2(l^2+m^2+n^2)-4\Omega^2n^2 \tag{54}$$

in this problem. For a steady disturbance ($\sigma_0=0$), equation (9) for the wave-number surface becomes

$$U^2n^2(l^2+m^2+n^2)-4\Omega^2n^2=0. \tag{55}$$

The zero-frequency wave-number surface $S(0)$ given by (55) is shown in figure 9, and has evidently a strong resemblance to the Rossby-wave case illustrated in figure 1. It includes the sphere

$$l^2+m^2+n^2=(2\Omega/U)^2, \tag{56}$$

just as the circle (24) was included in the Rossby-wave case. Wave-numbers on this sphere correspond to waves of uniform length $\pi U/\Omega$ and arbitrary direction. The fact, at first sight surprising, that any such wave must remain stationary with respect to the moving disturbance, is well known to students of the literature, as also is their combination in various standard normal modes, e.g. of Bessel-function type. As in figure 1, the directions of the arrows on the sphere are such that these waves are found only behind the forcing region.

In a manner equally reminiscent of figure 1, the surface $S(0)$ in figure 9 includes a straight portion, here the plane $n = 0$. An essential point of difference, however, is that the surface $S(0)$ defined by the quartic equation (55) consists of the sphere

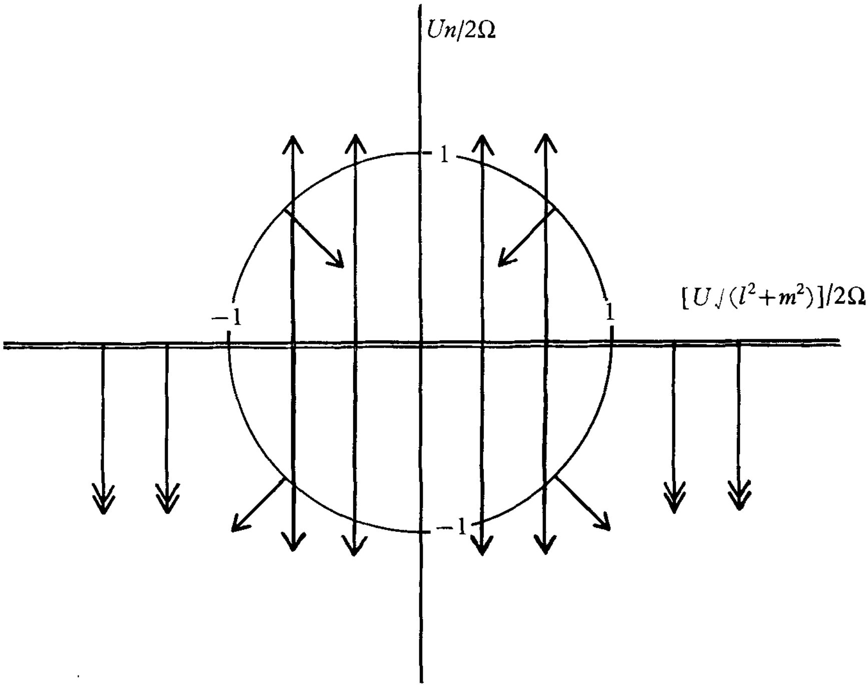

FIGURE 9. Wave-number surface for inertial waves generated by steady axial motion o an obstacle with velocity $(0, 0, U)$ through fluid rotating at angular velocity $(0, 0, \Omega)$. It consists of a sphere and two coincident planes.

(56) and the plane *taken twice*: it is a sphere and *two* (coincident) planes. It may be expected, therefore, that arrows along the appropriate normal must be drawn on both planes, and that the normal directions appropriate to each plane may or may not coincide.

The actual directions for waves with n (and σ) zero, that is, for disturbances independent of z (and t), are shown in figure 9. Disturbances whose transverse wave-number $\surd(l^2+m^2)$ exceeds $2\Omega/U$ trail behind the obstacle, because the arrows on *both* planes point in the negative z-direction, and those for which it is less than $2\Omega/U$ are found partly behind and partly in front, because the arrows on each plane point in opposite directions. This fact can be deduced in various ways, of which perhaps the easiest is actually to draw $S(\sigma)$ for various σ as in figure 10 below, and to observe that for small positive σ the plane $n = 0$ splits into two sheets, which lie on different sides of it where

$$\surd(l^2+m^2) < 2\Omega/U, \tag{57}$$

and otherwise lie both below it. For a more analytical deduction, see below.

The physical explanation of the result is that these waves with zero phase velocity, whose stationary crests are parallel to the axis of rotation, have a group velocity $2\Omega/\sqrt{(l^2+m^2)}$ directed along the axis of rotation (either up or down it). This exceeds U (so that forward influence becomes possible) if and only if (57) is satisfied. The waves propagate (for the inviscid fluid here treated) without attenuation, because the associated part of the wave-number surface is plane. After a long enough time, those for which (57) is satisfied extend arbitrarily far, both ahead of and behind the obstacle, in a 'Taylor column'.

Admittedly obstacles whose transverse dimension, say a, is small (so that the Rossby number

$$U/2\Omega a \tag{58}$$

is large) cannot significantly excite waves satisfying (57). But, as the ratio (58) decreases, transverse disturbances satisfying (57) can increasingly be excited by the obstacle. The disturbance that extends ahead of the obstacle is then the transverse disturbance created by the obstacle, modified as in §4 by a 'low-pass filter' passing only wave-numbers below $2\Omega/U$.

In contrast with §4, however, the disturbance that extends behind the obstacle is not subjected merely to the complementary high-pass filter; it includes, in fact, also some low-wave-number terms. To obtain an estimate of their magnitude, the method leading to equation (11) cannot be used without change because the integral to be estimated has a double-pole singularity on doubly covered portions of the wave-number surface. The modifications to the method that are needed are as follows.

With P as in (54) and σ_0 replaced by $i\epsilon$, equation (8) becomes

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i(lx+my)\}\,dl\,dm \int_{-\infty}^{\infty} \frac{F(l,m,n)\exp\{in(z-Ut)\}\,dn}{(Un+i\epsilon)^2(l^2+m^2+n^2)-4\Omega^2n^2}, \tag{59}$$

and the problem is to estimate the inner integral when $|z-Ut|$ is large. When ϵ is positive but very small, the double pole at $n=0$ is split into two simple poles at

$$n_1 = \frac{i\epsilon}{\dfrac{2\Omega}{\sqrt{(l^2+m^2)}}-U} \quad \text{and} \quad n_2 = \frac{i\epsilon}{\dfrac{2\Omega}{\sqrt{(l^2+m^2)}}+U}. \tag{60}$$

When (57) is satisfied these are on opposite sides of the real axis, so that by Jordan's lemma there is a contribution to the inner integral from the pole $n=n_1$ when $z-Ut$ is positive and from $n=n_2$ when it is negative; but, when (57) is not satisfied, both n_1 and n_2 have negative imaginary parts and so there is no contribution at all for $z-Ut>0$; this agrees with the direction of the arrows in figure 9.

More precisely, when (57) is satisfied, calculation of the residues at the poles gives a contribution to the inner integral in each case of

$$-\frac{\pi}{2\Omega\epsilon\sqrt{(l^2+m^2)}}F(l,m,n)\exp\{in(z-Ut)\}, \tag{61}$$

where $n=n_1$ for $z-Ut>0$ and $n=n_2$ for $z-Ut<0$. When (57) is not satisfied, both contributions appear for $z-Ut<0$ (and none for $z-Ut>0$) but that from $n=n_1$ has its sign changed.

Particularly when (57) *is* satisfied, there is apparently a difficulty in taking the limit of expressions (61) as $\epsilon \to 0$. The difficulty disappears, however, when we realize that in this problem, unlike the Rossby-wave case (see footnote in §5), F is necessarily zero for $n = 0$. To see this, we note that the basic equation (53) was obtained (S, §6) by applying the operation $(\partial/\partial t)$ curl to Helmholtz's equation for the vorticity in its linearized form S, (23). Hence, if Helmholtz's equation is transformed by a travelling forcing effect into a form with a forcing term,

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{v} = 2\Omega\frac{\partial \mathbf{v}}{\partial z} + \mathbf{g}(x, y, z - Ut), \tag{62}$$

equation (53) becomes

$$\frac{\partial^2}{\partial t^2}\nabla^2\mathbf{v} + 4\Omega^2\frac{\partial^2\mathbf{v}}{\partial z^2} = -\left(\frac{\partial}{\partial t}\operatorname{curl} + 2\Omega\frac{\partial}{\partial z}\right)\mathbf{g}. \tag{63}$$

The right-hand side of (63) can be written $\mathbf{f}(x, y, z - Ut)$, if

$$\mathbf{f} = (\partial/\partial z)(U\operatorname{curl}\mathbf{g} - 2\Omega\mathbf{g}), \tag{64}$$

whose Fourier transform $\mathbf{F}(l, m, n)$ evidently contains a factor n and so vanishes at $n = 0$.

The limit of (61) (which represents, as we saw, the inner integral in the expression (59) for ϕ) is found simply, therefore, by de l'Hôpital's rule, and is

$$-\frac{\pi i(\partial F/\partial n)_{n=0}}{2\Omega[2\Omega \mp U\sqrt{(l^2+m^2)}]}, \tag{65}$$

with the upper sign in the limit for $n = n_1$ and the lower in the limit for $n = n_2$. Disturbances are found ahead of the obstacle only when (57) is satisfied, and furthermore are only the n_1 disturbances, for which the negative sign taken in (65). We see that such disturbances are progressively amplified as $\sqrt{(l^2+m^2)}$ increases towards the limit (57), even though no disturbance at all appears ahead of the obstacle beyond that limit.

The physical reason why a forcing effect of given axial extent (represented by its first moment $(\partial F/\partial n)_{n=0}$) can excite a forward-moving wave component most powerfully when its group velocity only slightly exceeds the speed of travel of the forcing effect is that the time available before the wave component escapes from the forcing region is then greatest. Increase of (65) to extremely large values would be restricted, however, by dissipation, by non-linearity, or by finiteness of duration of the forcing effect.

For low values of the Rossby number (58), the obstacle can generate substantial disturbances satisfying (57). In the steady state these will be found, as (65) indicates, with a greater amplitude ahead of the obstacle than behind it, in agreement with the experiments of Taylor (1922, 1923) and Long (1953). Although theoretical work on this problem took time to catch up with experiment, the need for such a disturbance ahead of the obstacle was clearly argued by Stewartson (1958).

When the fluid, instead of being unbounded, is contained within a circular cylinder of radius b, whose axis is the axis of rotation along which the obstacle

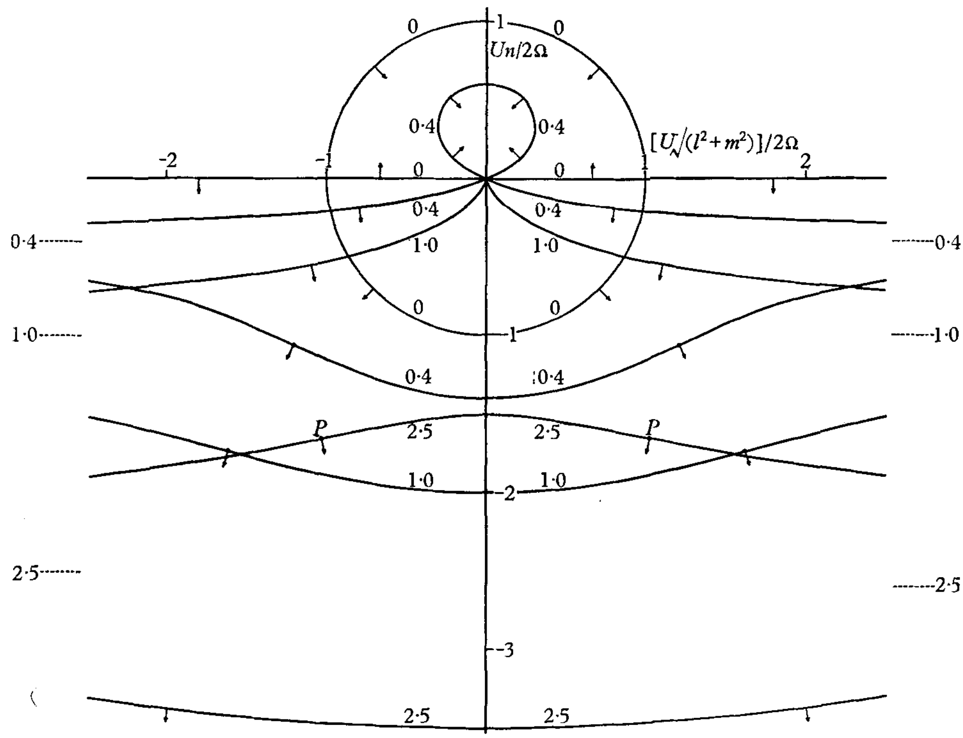

FIGURE 10. Wave-number surfaces for inertial waves generated by axial motion, with velocity (0, 0, U), of an oscillatory forcing effect with various frequencies σ_0, through fluid rotating at angular velocity (0, 0, Ω). The numbers 0, 0·4, 1·0 and 2·5 on the curves give the value of $\sigma_0/2\Omega$ in each case. - - - - - -, asymptotes.

moves, the spherical waves satisfying (56) have to be combined in axisymmetrical normal modes (wave-guide modes of Bessel-function type) satisfying the boundary condition on the cylindrical surface. The theory states, in agreement with the experiments of Long (1953), that these waves appear only behind the obstacle. However, a disturbance independent of z is found ahead of the obstacle provided that disturbances with wave-numbers satisfying (57) can be combined into a solution satisfying the boundary condition. This requires that

$$U/2\Omega b < j_1^{-1} = 0{\cdot}261,$$

where j_1 is the least positive zero of the Bessel function J_1, in agreement with arguments of Trustrum (1964).

Nigam & Nigam (1962) applied the methods of Lighthill (1960) (essentially those of this paper) to the more general case of waves made by a periodic forcing effect with frequency σ_0 moving along the axis of rotation in unbounded fluid. For example, an oscillating obstacle would make such waves, normally in addition to those that would be generated by its steady motion.

The surface $S(\sigma_0)$, which is now singly covered, has the equation

$$(\sigma_0 + Un)^2(l^2+m^2+n^2) - 4\Omega^2 n^2 = 0, \tag{66}$$

and figure 10 shows its shape for various values of the ratio $\sigma_0/2\Omega$. When $\sigma_0/2\Omega < 1$, the arrows indicating directions in which waves will be found show that they are still present ahead of the obstacle but that a cone of semi-angle $\sin^{-1}(\sigma_0/2\Omega)$ is empty of such waves. Furthermore, the limits on their wave-number are even more restrictive than in the zero-frequency case (57). By contrast, waves of larger wave-number, corresponding to the two sheets of the wave-number surface below the plane $n = 0$, trail behind the obstacle inside a cone, also of angle $\sin^{-1}(\sigma_0/2\Omega)$.

When $\sigma_0/2\Omega > 1$ there are only trailing waves, again confined within a cone. The point P on the wave-number surface corresponds to the waves found on the boundary of this cone. Nigam & Nigam (1962) give the surfaces of constant phase (reciprocal polars of $S(\sigma_0)$) for particular values of the ratio $\sigma_0/2\Omega$. These have cusps corresponding to any point of inflexion (such as P) on $S(\sigma_0)$.

9. Motion of obstacle perpendicular to axis of rotating fluid

The wave systems discussed in §§4, 5 and 6 were two-dimensional, while three-dimensional wave systems were studied in §§7 and 8 only for problems with axial symmetry. We conclude the paper, however, by discussion of a genuine three-dimensional problem.

An obstacle moving steadily at small Rossby number perpendicular to the axis of a uniformly rotating homogeneous fluid (rather than, as in §8, along it) can, as is well known, set in motion a 'Taylor column' of fluid, approximately cylindrical in shape with generators parallel to the axis, and moving with the obstacle at right angles to the axis. Experimental work on this subject has used, on the whole, somewhat limited volumes of fluid, and it remains uncertain how far along the axis the Taylor column extends in practice.

From the theoretical point of view, two limitations on its extent would be expected, viscous (non-zero Ekman number) and inertial (non-zero Rossby number). In different situations either of these may dominate. Morton (1966) has studied the limitation due to viscosity for zero Rossby number. Here the limitation due to non-zero Rossby number will be studied for an inviscid fluid.

For non-zero Rossby number it cannot be supposed that the Taylor column extends all the way to infinity, and indeed at very large distances from the body disturbances must be supposed to become small, and therefore subject to linear analysis. Moreover, even if this assumption were false, the linear analysis of the far field which follows would, according to the precedents of §§4 and 8, be expected to show up any possible propagation without amplitude reduction.

Accordingly, the near field is regarded as a travelling steady forcing effect which, in the far field, generates small disturbances. These must take the form of inertial waves, for which $P(\sigma, l, m, n)$ is given by (54). If the velocity of travel is $(U, 0, 0)$ then equation (9) for the wave-number surface $S(0)$ appropriate to a steady disturbance takes the form

$$U^2l^2(l^2+m^2+n^2)-4\Omega^2n^2 = 0. \tag{67}$$

Figure 11 indicates the shape of the surface (67) by plotting the contours m = constant in the (l, n) plane. The arrows normal to the surface pointing towards $S(+\delta)$ are easily shown to trail (as in §7) behind the direction of motion of the forcing region; their projections on to the (l, n) plane are shown in figure 11.

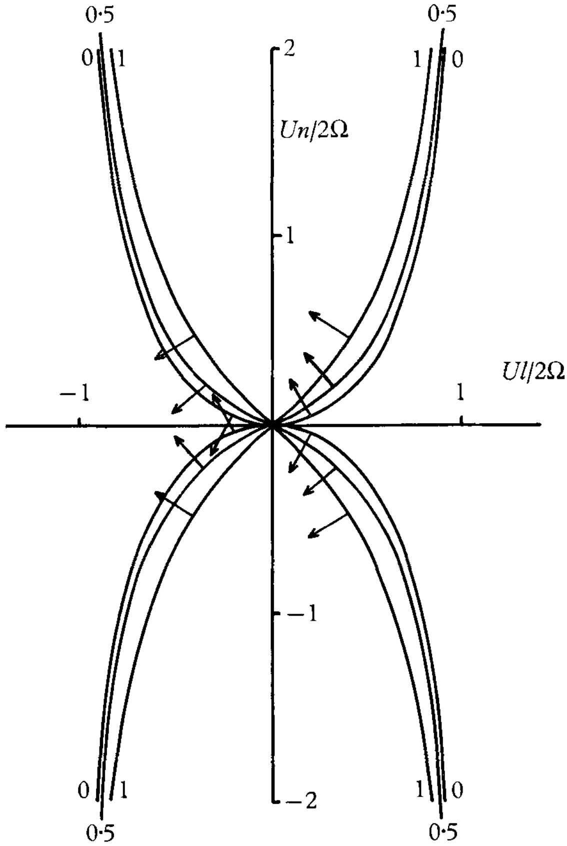

FIGURE 11. Wave-number surface (illustrated by means of contours of constant m in the (l, n) plane) for steady transverse motion of an obstacle with velocity $(U, 0, 0)$ through fluid rotating at angular velocity $(0, 0, \Omega)$. Contours for constant values 0, 0·5 and 1 (marked on the curves) of the ratio $Um/2\Omega$ are shown.

An obstacle of dimension a whose Rossby number, given by (58), is small generates waves whose characteristic wave-number k satisfies

$$0 \leqslant Uk/2\Omega \leqslant \epsilon, \tag{68}$$

where ϵ is a typical maximum Rossby number of the waves generated and would be expected to be proportional to (58). Accordingly, only the part of figure 11 which lies within a sphere of radius ϵ and centre the origin corresponds to waves which that obstacle can generate.

Within that part of the surface, the arrows all make a small angle with the z-direction, in agreement with the idea that to a first approximation the disturbance does not vary with z (as in the Taylor column). However, by studying their departure from the z-direction, we can investigate quantitatively how the disturbance trails behind the obstacle.

It follows from (61) and (62) that

$$\left|\frac{n}{l}\right| = \frac{Uk}{2\Omega} \leqslant \epsilon \quad \text{and} \quad \left|\frac{Ul}{2\Omega}\right| \leqslant \frac{Uk}{2\Omega} \leqslant \epsilon, \tag{69}$$

and using the fact that both these quantities are small the direction of the normal to (67) may be approximated as

$$\left(-\frac{Uk}{2\Omega}(\tfrac{3}{2}+\tfrac{1}{2}\cos 2\phi),\ -\frac{Uk}{2\Omega}(\tfrac{1}{2}\sin 2\phi),\ \pm 1\right), \tag{70}$$

where ϕ is defined by the equations

$$\frac{l}{\sqrt{(l^2+m^2)}} = \cos\phi, \quad \frac{m}{\sqrt{(l^2+m^2)}} = \sin\phi. \tag{71}$$

For fixed $Uk/2\Omega$ the directions (70) fill two cones whose axes have directions $(-\frac{3}{2}(Uk/2\Omega), 0, \pm 1)$ and whose semi-angles are $\frac{1}{2}(Uk/2\Omega)$.

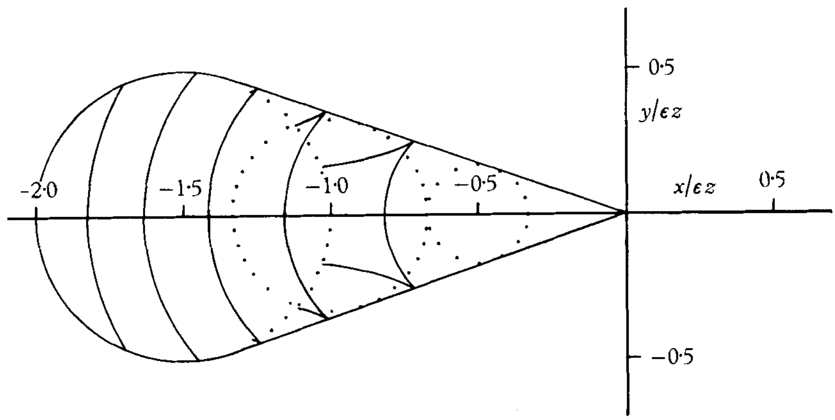

FIGURE 12. Transverse steady motion of an obstacle with velocity $(U, 0, 0)$ through fluid rotating at angular velocity $(0, 0, \Omega)$ produces at large axial distances z from the obstacle a region of waves (plain curves) whose cross-section is as shown. Dotted lines: loci of constant wavelength.

Waves are found, then, only in such cones trailing behind the obstacle, where $Uk/2\Omega$ takes all values from 0 to ϵ. The region filled by such cones, where waves are found, is shown (in cross-section by a plane z = constant) in figure 12. The dotted circles (cross-sections of the above-mentioned cones) are loci of fixed wave-number k.

Within the region in figure 12, the shape of the surfaces of constant phase is given parametrically by equation (15), with $\sigma_0 = 0$. This gives

$$\frac{x}{\sqrt{|AUz/2\Omega|}} = -\frac{\frac{3}{2}+\frac{1}{2}\cos 2\phi}{\sqrt{|\cos\phi|}}, \quad \frac{y}{\sqrt{|AUz/2\Omega|}} = -\frac{\frac{1}{2}\sin 2\phi}{\sqrt{|\cos\phi|}}, \quad |\cos\phi| \geqslant \frac{AU}{2\Omega z\epsilon^2}, \tag{72}$$

and a few such surfaces, with equal phase difference between each, are shown (again in cross-section by a plane z = constant) in figure 12. Those that extend to the straight lines $|y/x| = 2^{-\frac{3}{2}}$, which form part of the boundary of the region within which waves are found, have cusps thereon.

Only order-of-magnitude estimates can be given concerning the matching of the far-field behaviour depicted in figure 12 with the near-field 'Taylor-column' behaviour. The cross-section in figure 12 has dimension of order $\epsilon|z|$ and can be expected to match with a Taylor column whose cross-section has dimension a in some transition region situated around $|z| = a/\epsilon$. This indicates that Taylor columns extend for a distance of order the dimension of the obstacle divided by the Rossby number (except in circumstances when viscosity is large enough to limit them to a smaller length).

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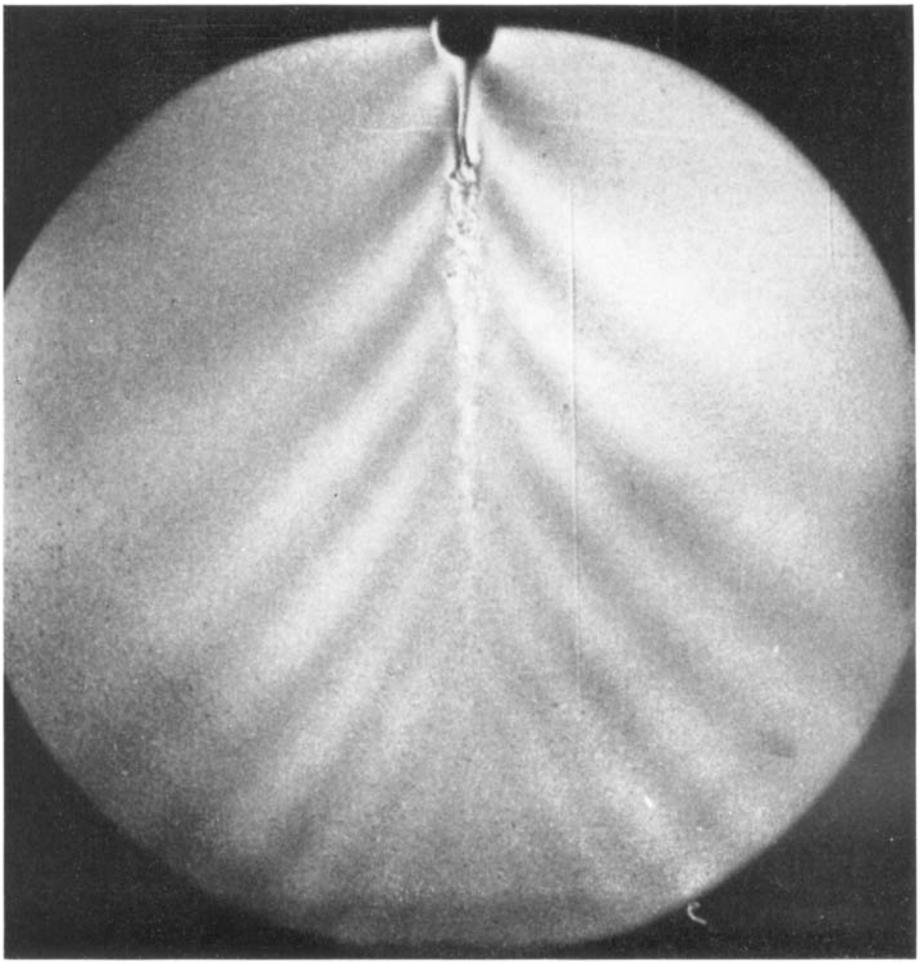

FIGURE 7. Schlieren photograph (Mowbray 1966) of waves generated by a sphere of diameter 2·54 cm rising vertically at a speed of 1·02 cm/s in a solution of sodium chloride whose density falls with height at a rate of 0·0020 gm/cm^3 per cm.